SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2016

Assessment Task #2

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Leave your answers in the simplest exact form, unless otherwise stated.
- In Questions 7-9, show relevant mathematical reasoning and/or calculations
- Start each **NEW** question in Section II in a separate answer booklet.

Total Marks – 88

Section I Pages 2–3

6 Marks

- Attempt Questions 1–6
- Allow about 10 minutes for this section.

Section II Pages 4–11

82 marks

- Attempt Questions 7–9
- Allow about 1 hour and 50 minutes for this section

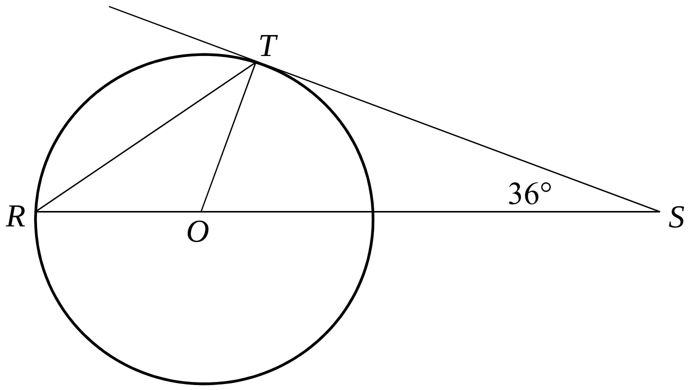
Examiner: *R Dowdell*

Section I

6 marks
Attempt Questions 1-6
Allow about 10 minutes for this section

Use the multiple-choice answer sheet for Questions 1-6.

1 ST is a tangent to the circle, centre O.
What is the size of $\angle TRS$?



A 13.5°

B 27°

C 36°

D 54°

2 What is the argument of the complex number $\dfrac{\sqrt{3}}{1+i\sqrt{3}}$?

A $\tan^{-1}\sqrt{3}$

B $-\tan^{-1}\sqrt{3}$

C 0

D $\pi-\tan^{-1}\sqrt{3}$

3 What are the square roots of $-2i$?

A $-1+i, \quad 1-i$

B $1+i, \quad -1-i$

C $-1-i, \quad -1+i$

D $1+i, \quad 1-i$

4 What are the zeros of the polynomial function

$f(x) = 2x^3 - 8x^2 + 6x$?

A 0, 1, 3

B 1, 2, 3

C 0, –1, –3

D 0, 1, –4

5 An electrical panel has five switches. How many ways can the switches be positioned, up or down, if three switches must be up and two must be down?

A 10

B 24

C 48

D 120

6 A research team of 6 people is to be formed from 10 chemists, 5 politicians, 8 economists and 15 biologists. How many possible teams can be formed with at least 5 chemists?

A 6772

B 6934

C 7266

D 8123

End of Section I

Section II

82 marks
Attempt Questions 7-9
Allow about 1 hour and 50 minutes for this section

Answer each question in a separate writing booklet. Extra writing booklets are available.

In Questions 7-9, your responses should include relevant mathematical reasoning and/or calculations.

Question 7 (28 marks) Start a NEW booklet

(a) The zeros of $x^3 - 2x^2 - 5x + 6$ are α, β and γ. **2**
Find a cubic polynomial, with integer coefficients, whose zeros are

$$\alpha - 1,\ \beta - 1 \text{ and } \gamma - 1$$

(b) (i) Use de Moivre's Theorem to express $\cos 4\theta$ in terms of $\cos\theta$. **2**

(ii) Use the result from part (i) to solve the equation $8x^4 - 8x^2 + 1 = 0$. **2**

(iii) Deduce that $\cos\frac{\pi}{8} \times \cos\frac{3\pi}{8} = \frac{1}{2\sqrt{2}}$. **2**

(c) Let $P(x) = x^5 - 5x^4 + ax^3 + bx^2 + cx - 8$, where a, b and c are real numbers.

(i) Find the values of a, b and c given that 2 is a zero of multiplicity 3. **4**

(ii) Factorise $P(x)$ into real linear and quadratic factors. **2**

Question 7 continues on page 5

Question 7 (continued)

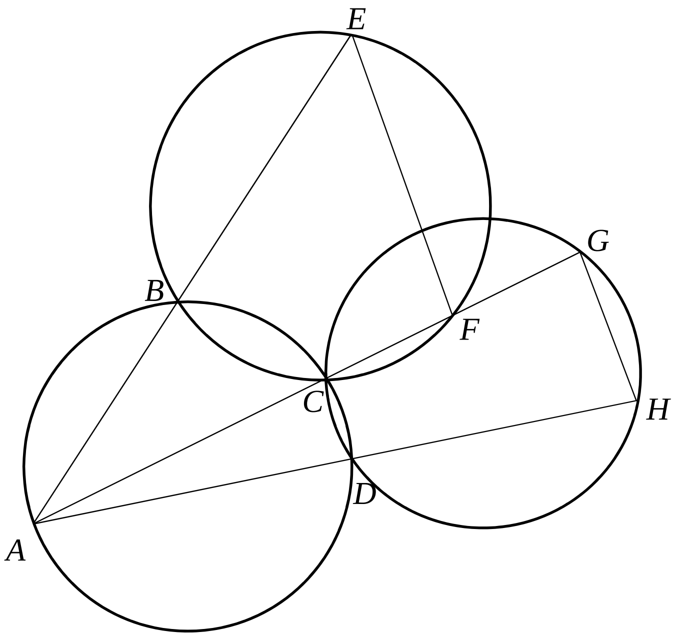
(d) (i) The polynomial $P(x) = ax^4 + bx^3 + cx^2 + dx + e$ has real coefficients. If α is a zero of $P(x)$, show that $\overline{\alpha}$ is also a zero. **2**

(ii) The polynomial $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$ has zeros $a + ib$ and $a + 2ib$, where a and b are real. **3**

Find a and b and then express $P(x)$ as the product of quadratic polynomials with real coefficients.

Use the supplied custom Writing Booklet for parts (e), (f) and (g)

(e)

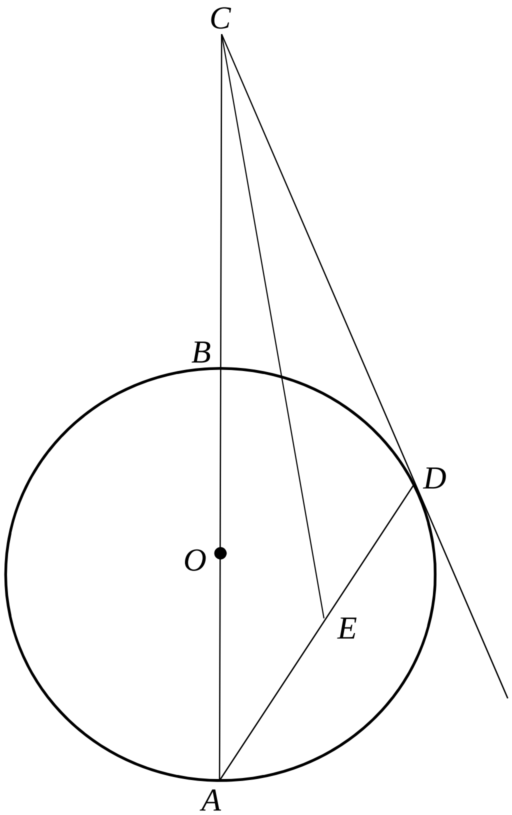


Prove that EF is parallel to GH. **3**

Question 7 continues on page 6

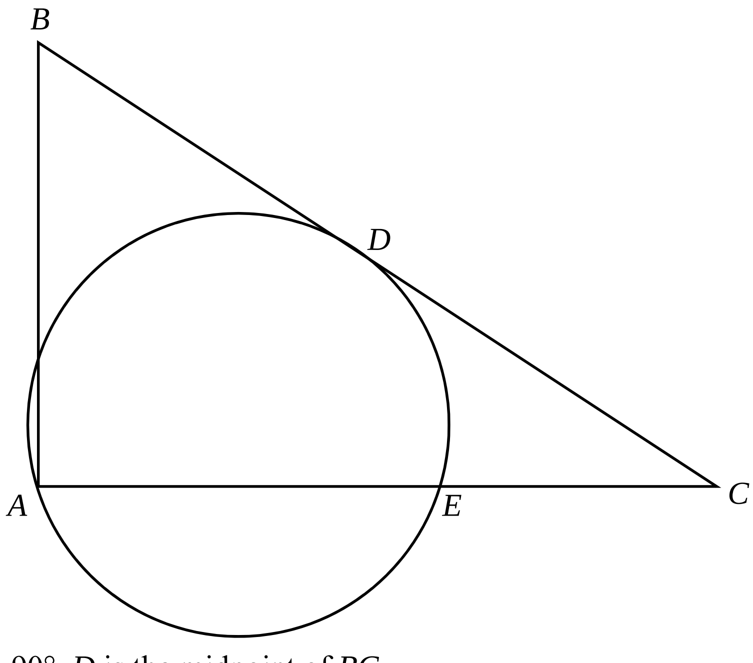
Question 7 (continued) **Use the supplied custom Writing Booklet for parts (e), (f) and (g)**

(f)



O is the centre of the circle and CE bisects $\angle OCD$. CD is a tangent. **3**
Prove that $\angle CED = 45°$.

(g)



In ΔABC, $\angle BAC = 90°$. D is the midpoint of BC.
A circle touches BC at D, passes through A and cuts AC again at E.

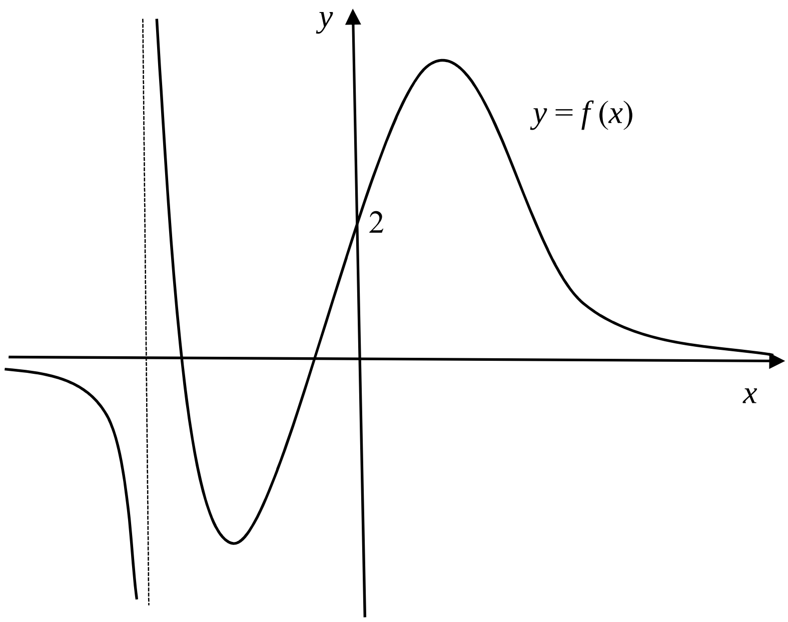
Prove that the length of arc $AD = 2 \times$ the length of arc DE. **3**

End of Question 7

Question 8 starts on Page 8

Question 8 (27 marks) **Use the supplied custom Writing Booklet**

(a)



If the graph represents $y = f(x)$, draw neat sketches of the following in your answer booklet:

(i) $y = (f(x))^2$ **2**

(ii) $y = |f(x)|$ **2**

(iii) $y = \sqrt{f(x)}$ **2**

(iv) $y = \dfrac{1}{f(x)}$ **2**

(v) $y = e^{f(x)}$ **2**

(vi) $y = f(e^x)$ **2**

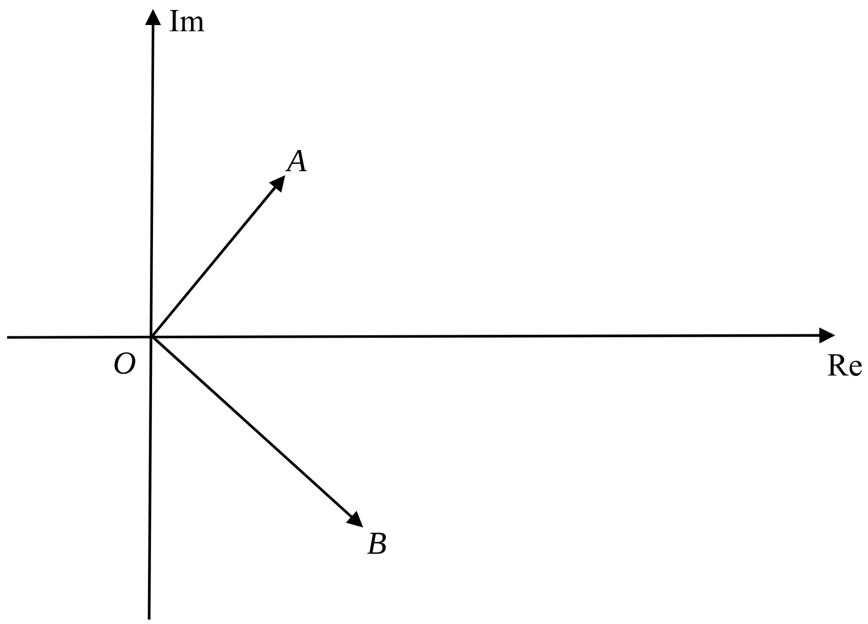
(vii) $y = f'(x)$ **2**

(viii) $y = \int f(x)\,dx$, where $y = 0$ when $x = 0$. **2**

Question 8 continues on Page 9

Question 8 (continued)

(b)



OB is perpendicular to OA and $OB = 2\ OA$. If the point A corresponds to the complex number z, what complex number corresponds to C, the midpoint of AB? **2**

(c) Sketch the region in the Argand diagram that satisfies the inequality $z\overline{z} - 3(z + \overline{z}) \leq 0$. **3**

(d) (i) Forty people are travelling to the Head of the River. Four can travel in a car, eight can travel in a minibus and the other twenty-eight in a large bus. (Each vehicle has been supplied with a driver.) In how many ways can the people be divided between the vehicles so that they can travel to the Head of the River? (Leave your answer in unsimplified form.) **2**

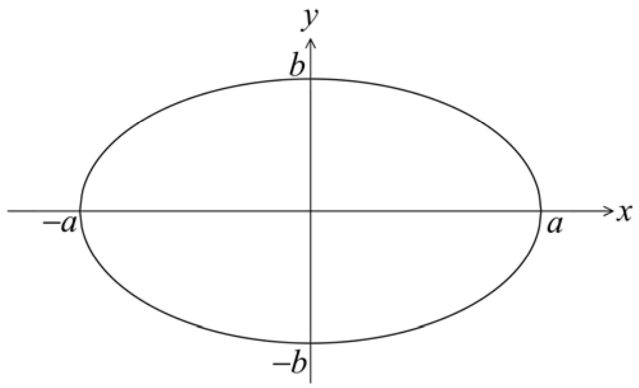
(ii) Suppose that the three groups have been chosen. The people who travelled by car wear white caps, the people who travelled on the minibus wear brown caps and the people who travelled on the large bus wear blue caps. When the people arrive at the Head of the River, they sit on a bench such that the people who travelled in each vehicle sit as a group. In how many ways can the people be arranged on the bench? (Leave your answer in unsimplified form.) **2**

(e) 6 people stay at a motel with 6 rooms. If the 6 people choose a room randomly, find the probability that no rooms are empty. **2**

End of Question 8

Question 9 (27 marks) **Start a NEW Writing Booklet**

(a) The graph of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is drawn below. **3**



Use the integral $\int_0^a \sqrt{a^2-x^2}\,dx$ to show that the area of the ellipse is πab.

(b) Consider $I=\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$

(i) Rewrite the integral using the substitution $x=\frac{\pi}{2}-y$. **2**

(ii) Hence show that $I=\frac{\pi}{4}$. **1**

(c) Consider $f(x)=x^n e^{-x}$.

(i) Find $f'(x)$. **2**

(ii) Find and classify any stationary points for $y=f(x)$, where $x\neq 0$. **2**

(iii) Use part (ii) to assist in explaining why $\lim_{x\to\infty} x^n e^{-x}=0$ for $n\geq 0$. **1**

(iv) The Gamma Function, $\Gamma(n)$, is defined as $\Gamma(n)=\int_0^\infty x^{n-1}e^{-x}\,dx$. **2**

Show that $\Gamma(n)=(n-1)\Gamma(n-1)$.

(Note that $\int_0^\infty x^{n-1}e^{-x}\,dx=\lim_{M\to\infty}\int_0^M x^{n-1}e^{-x}\,dx$.)

(v) Show that $\Gamma(n)=(n-1)!$ for integral $n>0$. **2**
You are NOT required to use mathematical induction.

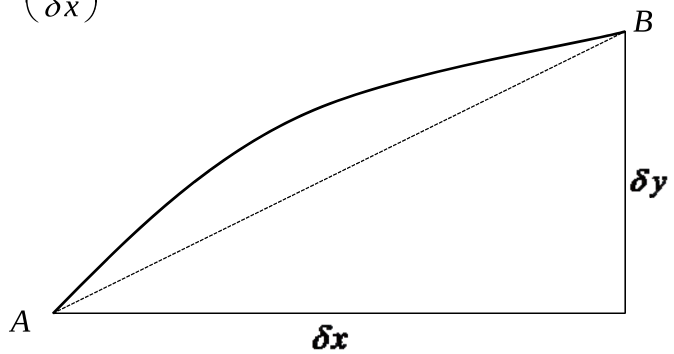
Question 9 continues on page 11

Question 9 (continued)

(d) (i) Show that $\int \sin^n x\,dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$. **4**

(ii) Evaluate $\int_0^{\frac{\pi}{3}} \sin^3 x\,dx$. **2**

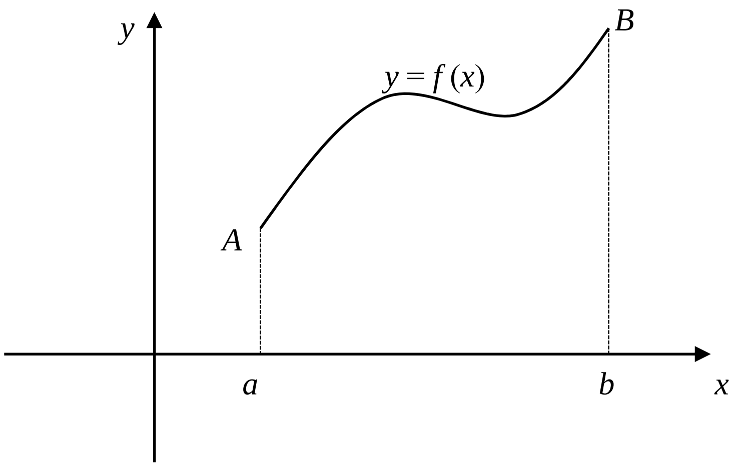
(e) (i) Show, using Pythagoras' Theorem, that an approximation to the length of arc *AB* is $\sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2}\,\delta x$. **2**



(ii) The length of the arc *AB* is given by **4**

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2}\,\delta x = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Verify, using integration, that the length of the semicircle $y=\sqrt{4-x^2}$ is 2π.



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Assessment Task #2

Mathematics Extension 2

Suggested Solutions

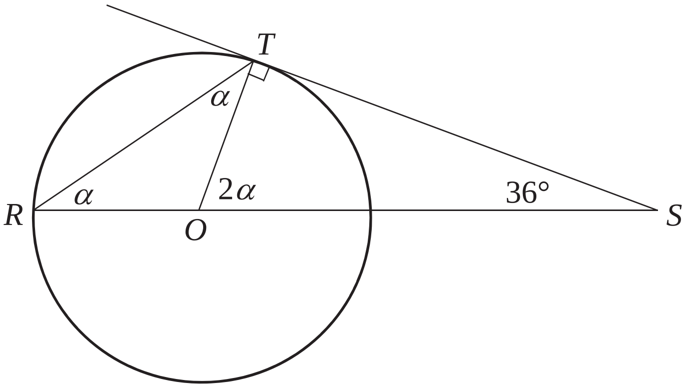
Q	Marker
7	AF
8	PB
9	PP

MC Answers

Q1	B
Q2	B
Q3	A
Q4	A
Q5	A
Q6	C

Section I MC Solutions

1 ST is a tangent to the circle, centre O.
What is the size of $\angle TRS$?



A 13.5°
(B) 27°
C 36°
D 54°

Let $\angle TRS = \alpha°$
$\therefore \angle TOS = 2\alpha°$ (angles at centre and circumference)
$\angle OTS = 90°$ (ST tangent)
$\therefore 2\alpha + 36 = 90$ (angle sum $\Delta\, OTS$)
$\therefore \alpha = 27$
$\angle TRS = 27°$

2 What is the argument of the complex number $\dfrac{\sqrt{3}}{1+i\sqrt{3}}$?

A $\tan^{-1}\sqrt{3}$
(B) $-\tan^{-1}\sqrt{3}$
C 0
D $\pi - \tan^{-1}\sqrt{3}$

$$\arg\left(\frac{\sqrt{3}}{1+i\sqrt{3}}\right) = \arg\left(\sqrt{3}\right) - \arg\left(1+i\sqrt{3}\right)$$
$$= 0 - \tan^{-1}\sqrt{3}$$
$$= -\tan^{-1}\sqrt{3}$$

3 What are the square roots of $-2i$?

(A) $-1 + i, \quad 1 - i$
B $1 + i, \quad -1 - i$
C $-1 - i, \quad -1 + i$
D $1 + i, \quad 1 - i$

$$[\pm(1-i)]^2 = 1 - 2i + i^2 = -2i$$

4 What are the zeros of the polynomial function $f(x) = 2x^3 - 8x^2 + 6x$?

(A) 0, 1, 3

B 1, 2, 3

C 0, –1, –3

D 0, 1, –4

$$\begin{aligned} f(x) &= 2x^3 - 8x^2 + 6x \\ &= 2x(x^2 - 4x + 3) \\ &= 2x(x-3)(x-1) \end{aligned}$$

5 An electrical panel has five switches. How many ways can the switches be positioned, up or down, if three switches must be up and two must be down?

(A) 10

B 24

C 48

D 120

How many words can be formed from UUUDD?

i.e. $\binom{5}{2} = 10$

6 A research team of 6 people is to be formed from 10 chemists, 5 politicians, 8 economists and 15 biologists. How many possible teams can be formed with at least 5 chemists?

A 6772

B 6934

(C) 7266

D 8123

Number of exactly 5 chemists + Number of all six chemists

$= \binom{10}{5} \times \binom{28}{1} + \binom{10}{6} = 7266$

End of Section I Solutions

7) a) let $X = x - 1$

$x = X + 1$

$(X+1)^3 - 2(X+1)^2 - 5(X+1) + 6$

$= X^3 + 3X^2 + 3X + 1 - 2(X^2 + 2X + 1) - 5X - 5 + 6$

$= X^3 + X^2 - 6X$

b) i) $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$ using de moivre's theorem

$(\cos\theta + i\sin\theta)^4 = c^4 + 4c^3(is) + 6c^2(is)^2 + 4c(is)^3 + (is)^4$

$= c^4 + 4c^3si - 6c^2s^2 - 4cs^3i + s^4$

equate real.

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta(1 - \cos^2\theta) + (1 - \cos^2\theta)^2$

$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$

$= 8\cos^4\theta - 8\cos^2\theta + 1$

ii) let $x = \cos\theta$

$8\cos^4\theta - 8\cos^2\theta + 1 = 0$

$\cos 4\theta = 0$

$4\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$\theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$

$x = \cos\frac{\pi}{8}, \cos\frac{3\pi}{8}, \cos\frac{5\pi}{8}, \cos\frac{7\pi}{8}$

iii) PRODUCT OF ROOTS:

$\cos\frac{\pi}{8} \cdot \cos\frac{3\pi}{8} \cdot \cos\frac{5\pi}{8} \cdot \cos\frac{7\pi}{8} = \frac{e}{a}$

$\cos\frac{\pi}{8} \cdot \cos\frac{3\pi}{8} \cdot \left(-\cos\frac{3\pi}{8}\right) \cdot \left(-\cos\frac{\pi}{8}\right) = \frac{1}{8}$

$\left(\cos\frac{\pi}{8}\cos\frac{3\pi}{8}\right)^2 = \frac{1}{8}$

$\cos\frac{\pi}{8} \cdot \cos\frac{3\pi}{8} = \pm\frac{1}{2\sqrt{2}}$

since $\frac{\pi}{8}$ & $\frac{3\pi}{8}$ are acute $\cos\frac{\pi}{8} \cdot \cos\frac{3\pi}{8} = \frac{1}{2\sqrt{2}}$

c). $P(x) = x^5 - 5x^4 + ax^3 + bx^2 + cx - 8$

let zeros be $2, 2, 2, \alpha, \beta$

$2 + 2 + 2 + \alpha + \beta = -\frac{B}{A}$

$6 + \alpha + \beta = \frac{-(-5)}{1}$

$\alpha + \beta = -1$

$2.2.2.\alpha.\beta = -\frac{F}{A}$

$8\alpha\beta = -\frac{(-8)}{1}$

$\alpha\beta = 1$

$x^2 + x + 1$ has zeros α & β.

$P(x) = (x-2)^3(x^2 + x + 1)$

which is the answer to (ii).

$P(x) = (x^3 + 3x^2(-2) + 3x(-2)^2 + (-2)^3)(x^2 + x + 1)$

$= (x^3 - 6x^2 + 12x - 8)(x^2 + x + 1)$

$= x^5 - 6x^3 + 12x^3 - 8x^2 + x^4 - 6x^3 + 12x^2 - 8x + x^3 - 6x^2 + 12x - 8$

$= x^5 - 5x^4 + 7x^3 - 2x^2 + 4x - 8$

$a = 7$, $b = -2$, $c = 4$

which is the answer to (i)

d) i) $P(\alpha) = a\alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha + e = 0$

$$\overline{a\alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha + e} = \overline{0}$$

$$\overline{a\alpha^4} + \overline{b\alpha^3} + \overline{c\alpha^2} + \overline{d\alpha} + \overline{e} = 0$$

$$a\overline{\alpha^4} + b\overline{\alpha^3} + c\overline{\alpha^2} + d\overline{\alpha} + e = 0$$

$$a(\bar{\alpha})^4 + b(\bar{\alpha})^3 + c(\bar{\alpha})^2 + d(\bar{\alpha}) + e = 0$$

$$P(\bar{\alpha}) = 0$$

$\therefore \bar{\alpha}$ is also a zero

ii) since $P(x)$ has real coefficients

$a+ib, a-ib, a+2ib, a-2ib$ are zeros of $P(x)$

SUM OF ROOTS:

$$a + ib + a - ib + a + 2bi + a - 2bi = -\frac{B}{A}$$

$$4a = \frac{-(-4)}{1}$$

$$\underline{a = 1}$$

PRODUCT OF ROOTS:

$$(a+ib)(a-ib)(a+2bi)(a-2bi) = \frac{E}{A}$$

$$(a^2+b^2)(a^2+4b^2) = \frac{10}{1}$$

$$(1+b^2)(1+4b^2) = 10$$

$$1 + 4b^2 + b^2 + 4b^4 = 10$$

$$4b^4 + 5b^2 - 9 = 0$$

$$b^2(4b^2+9) - (4b^2+9) = 0$$

$$(4b^2+9)(b^2-1) = 0$$

$$b^2 = 1 \qquad b^2 = -\frac{9}{4}$$

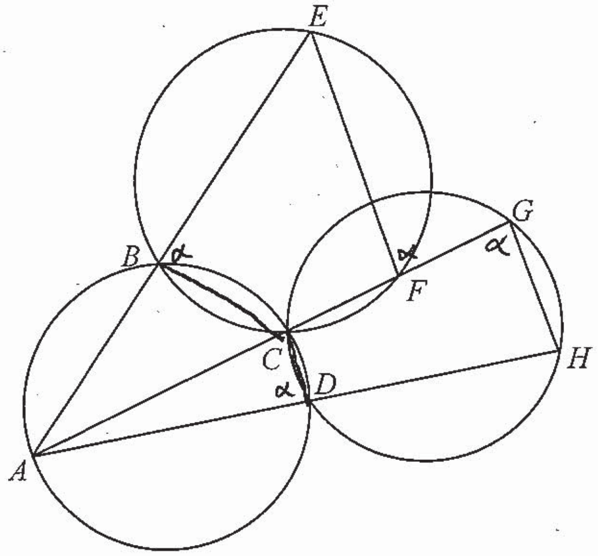
since b is real $\underline{b = \pm 1}$

$$P(x) = (x^2 - 2x + 2)(x^2 - 2x + 5)$$

SBHS 2016 Yr 12 Mathematics Extension 2

Answer sheet for Question 7 parts (e) – (g)   Student Number: ______________

(e)



let $\angle FGH = \alpha$

$\angle ADC = \alpha$ (exterior angle of a cyclic quadrilateral)

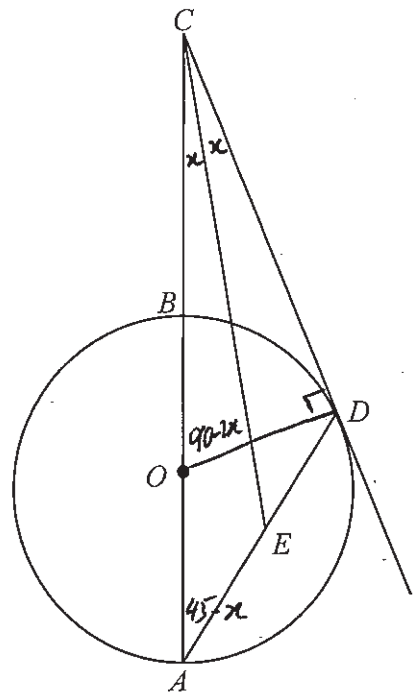
$\angle EBC = \alpha$ (as above)

$\angle EFG = \alpha$ (as above)

$\angle EFG = \angle FGH$

since alternate angles are equal $EF \parallel GH$.

(f)



$\angle ODC = 90°$ (radius ⊥ tangent)

let $\angle DCE = x$

$\angle ECA = x$ (given CE bisects $\angle OCD$)

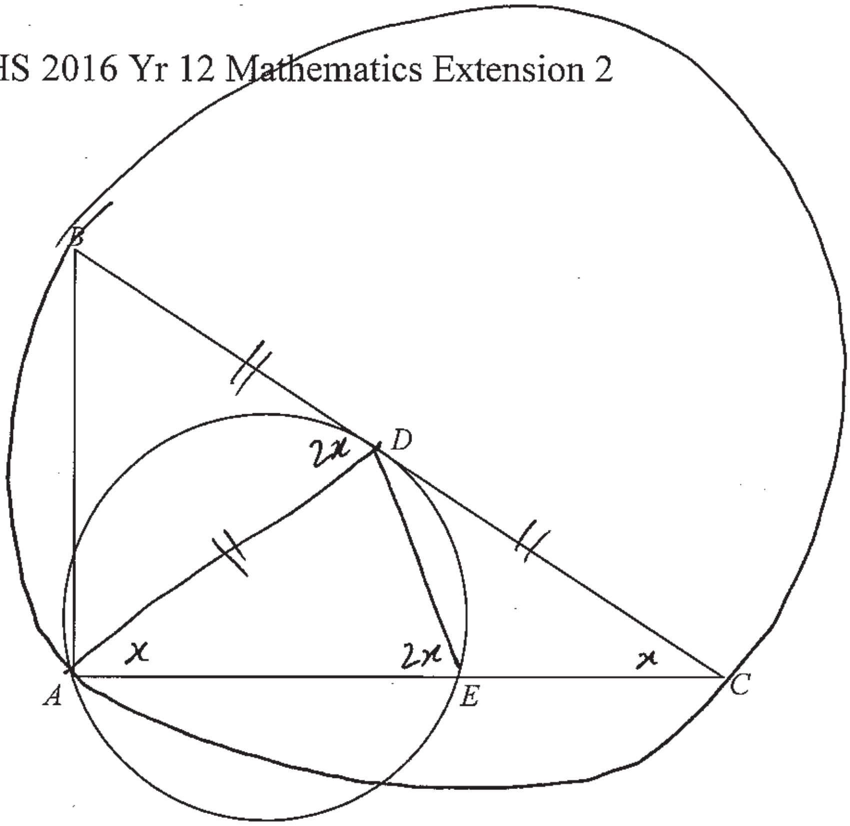
$\angle COD = 90 - 2x$ (angle sum of $\triangle COD$)

$\angle BAD = 45 - x$ (angle at centre is twice angle at circumference on arc BD)

$\angle CED = 45 - x + x$ (exterior angle of $\triangle ACE$)

$= 45°$

(g)



Draw circle through A, B, C

since $\angle BAC = 90°$, BC is diameter (converse of angle in semicircle)

since D is midpoint of BC

D is the centre

$DA = DC = DB$ (equal radii)

let $\angle DCA = x$

$\angle DAE = x$ (opposite equal sides, $\triangle DAC$)

$\angle BDA = 2x$ (exterior angle of $\triangle DAC$)

$\angle DEA = 2x$ (alternate segment theorem)

since ~~angle~~ subtended by arc AD is double the angle subtended by arc DE

arc $AD = 2 \times$ arc DE

COMMENTS:

7) a) Done well by students

b) ii) Care needed to be taken when choosing the values of θ

iii) $\cos\frac{5\pi}{8} = -\cos\frac{3\pi}{8}$ should be shown in working.

c) i) Many students had trouble solving 3 equations, 3 unknowns.

ii) Those students should have been aware of their error when answering this question.

d) i) Show that $\bar{\alpha}$ is also a zero.

Do not just state the conjugate root theorem

CIRCLE GEOMETRY

e) "supplementary angles" should not be given as a reason.

Why are two angles supplementary?
- angles on a line
- co-interior angles on parallel lines
- opposite angles of a cyclic quadrilateral.

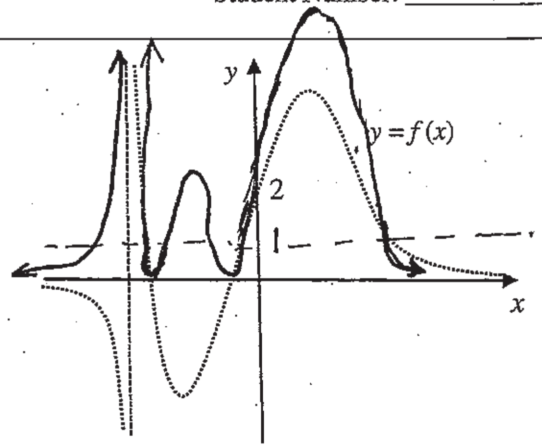
f) There are many ways of answering this question. An angle of 90° is needed. Either ∠BDA or ∠ODC.

g) Was not answered very well. Very few were awarded full marks.

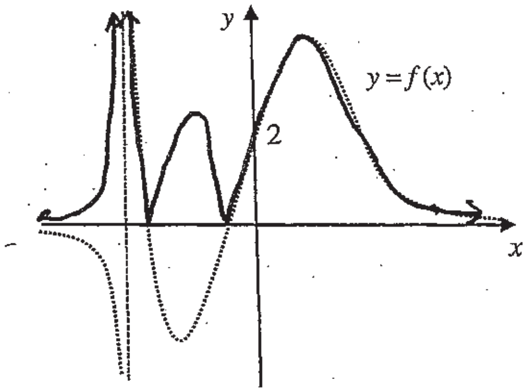
SBHS 2016 Yr 12 Mathematics Extension 2

Answer sheet for Question 8 Student Number: ____________________

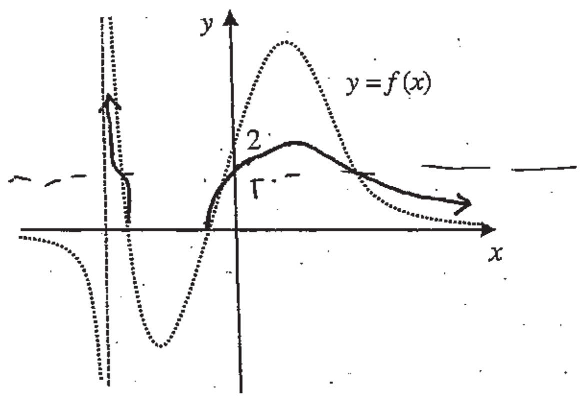
(a) (i) $y = (f(x))^2$



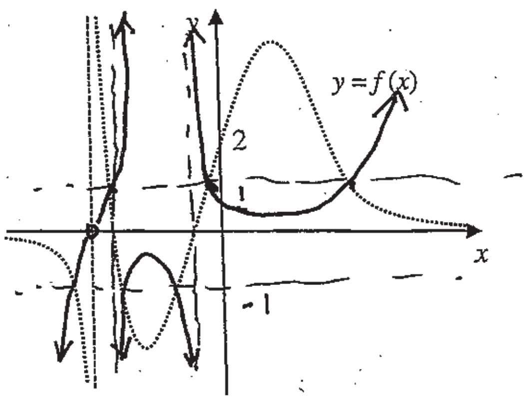
(a) (ii) $y = |f(x)|$



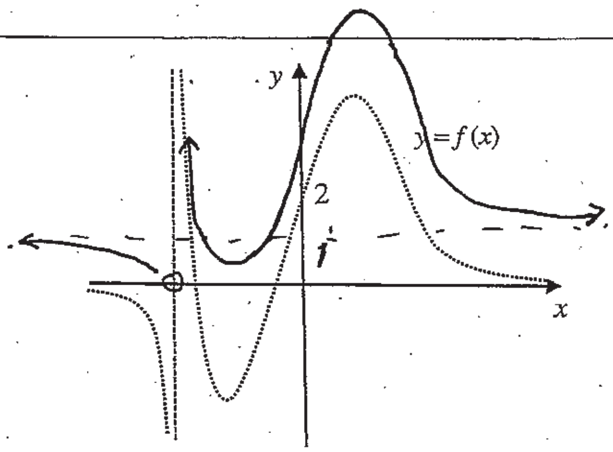
(a) (iii) $y = \sqrt{f(x)}$



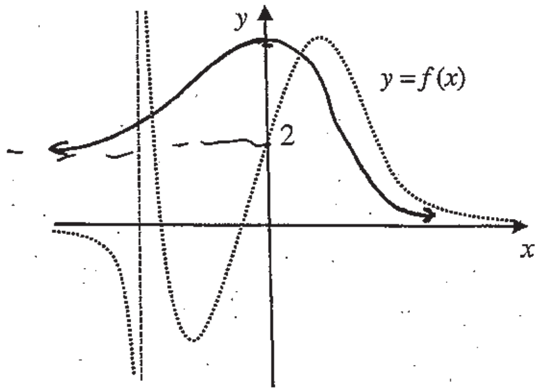
(a) (iv) $y = \dfrac{1}{f(x)}$



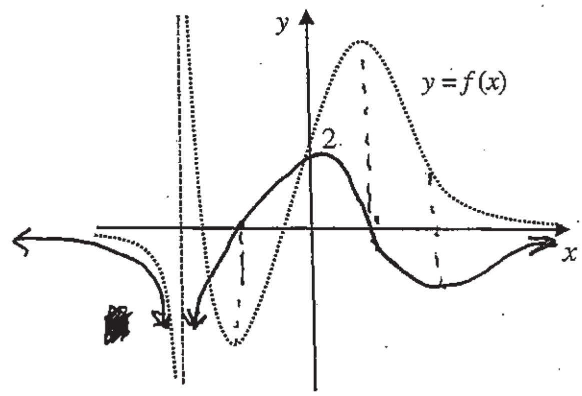
(a) (v) $y = e^{f(x)}$



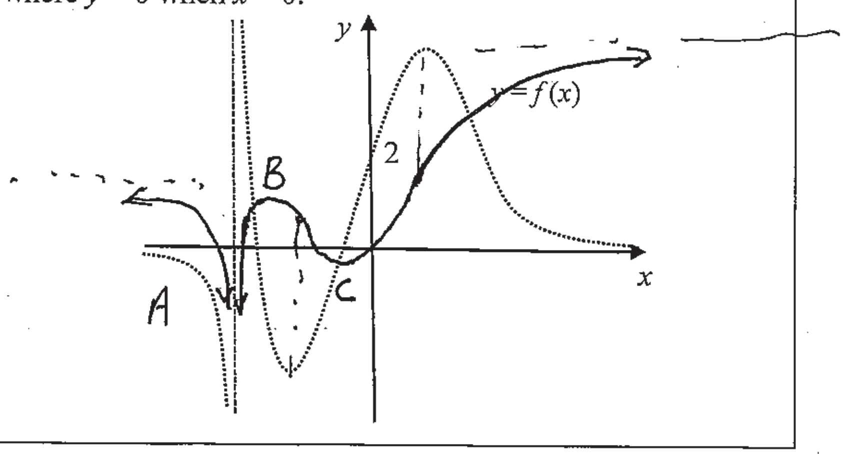
(a) (vi) $y = f(e^x)$



(a) (vii) $y = f'(x)$



(a) (viii) $y = \int f(x)\,dx$, where $y = 0$ when $x = 0$.



NB/ A could be a cusp.
B could be below the x-axis.
C has to be below the x-axis.
above the x-axis.

QUESTION 8 (X2)

(a) (i) – (viii) Each question is out of 2 marks. ½ mark deducted for each minor error. 1 mark deducted for each significant error.

COMMENT

(i) well done. – significance of $y=1$ was observed, as the case in the smooth turning points

(ii). well done.

(iii). $y=1$ was significant as was the vertical nature of the roots

(iv) well done.

(v). asymptote at $y=1$.

(vi) proved difficult. Asymptote at $y=2$ where $x<0$.

(vii) need to match turning points and roots. also inflexions and turning points.

(viii) see graph.

(b). A represents z

∴ B represents $-2iz$. [2]

∴ mid-pt of AB is $\frac{z-2iz}{2}$.

COMMENT $\frac{z+2\bar{z}}{2}$ and $\frac{(x+2y)+i(y-2x)}{2}$

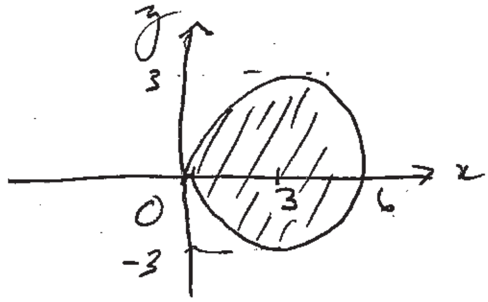
were correct answers.

(c) $z\bar{z} - 3(z+\bar{z}) \le 0$ [3]

∴ $x^2+y^2-3\times 2x \le 0$

$x^2-6x+9+y^2 \le 0$

$(x-3)^2+y^2 \le 0$



COMMENT well done

(d)(i) $\binom{40}{28} \times \binom{12}{8} \times \binom{4}{4}$ [2]

COMMENT well done.

Alternatively $\binom{40}{4} \times \binom{36}{8} \times \binom{28}{28}$ etc

(ii) $3! \times 28! \times 8! \times 4!$ OR $\frac{40!}{4! \times 8! \times 28!}$ [2]

COMMENT Some left out 3!

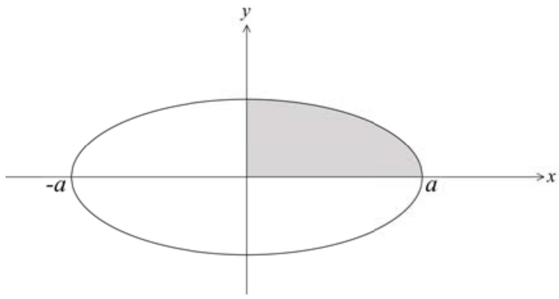
(e) $\frac{6!}{6^6} = \frac{5!}{6^5}$ [2]

COMMENT most were able to do this part.

Question 9 Solutions (27 marks)

(a) The graph of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is drawn below. **3**

Use the integral $\displaystyle\int_0^a \sqrt{a^2-x^2}\,dx$ to show that the area of the ellipse is πab.



$$\int_0^a \sqrt{a^2-x^2}\,dx=\tfrac{1}{4}\pi a^2$$ as it represents the area of a quarter of a circle.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\Rightarrow\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$\therefore y^2=\frac{b^2}{a^2}(a^2-x^2)$$

$$\therefore y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

To get the area of the ellipse, first find the area for $0\le x\le a$.

Area quarter ellipse = $\displaystyle\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$

$$=\frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$=\frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$=\frac{b}{a}\times\frac{1}{4}\pi a^2$$

$$=\frac{1}{4}\pi ab$$

So the area of the ellipse is πab.

Comment

Only 1 mark was available for $\displaystyle\int_0^a \sqrt{a^2-x^2}\,dx$, though not many students used the fact that it represented the area of a quadrant of a circle.

Students who tried to argue that $a = b$ and then try and get the area of the ellipse, got no credit.

Question 9 (continued)

(b) Consider $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$ **2**

(i) Rewrite the integral using the substitution $x = \frac{\pi}{2} - y$.

$x = \frac{\pi}{2} - y \Leftrightarrow y = \frac{\pi}{2} - x$

If $x = \frac{\pi}{2} - y$ then $dx = -dy$

$x = 0, y = \frac{\pi}{2}; x = \frac{\pi}{2}, y = 0$

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx \\ &= \int_{\frac{\pi}{2}}^{0} \frac{\sqrt{\sin(\frac{\pi}{2} - y)}}{\sqrt{\sin(\frac{\pi}{2} - y)} + \sqrt{\cos(\frac{\pi}{2} - y)}} (-dy) \\ &= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos y}}{\sqrt{\cos y} + \sqrt{\sin y}}\, dy \\ &= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx \end{aligned}$$

Comment

Some students wanted to make use of the theorem: $\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$.

No credit was given if they didn't quote it.

(ii) Hence show that $I = \frac{\pi}{4}$. **1**

$$\begin{aligned} 2I &= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx + \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx \\ &= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx \\ &= \int_0^{\frac{\pi}{2}} 1\, dx \\ &= \left[x \right]_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{2} \end{aligned}$$

$\therefore I = \frac{\pi}{4}$

Alternative:
Some students made use of the fact that

$$1 - \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} = \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}$$

Comment:
This problem was generally well done as most students knew the trick of adding the two integrals

Question 9 (continued)

(c) Consider $f(x) = x^n e^{-x}$, $n > 0$.

(i) Find $f'(x)$. **2**

$f'(x) = nx^{n-1}e^{-x} + (-e^{-x})x^n$

$= x^{n-1}e^{-x}(n-x)$

Comment:
Students who factorised fully here, generally went on to success in the next few parts.

(ii) Find and classify any stationary points for $y = f(x)$, where $x \neq 0$. **2**

Stationary points occur when $f'(x) = 0$.

$\therefore x^{n-1}e^{-x}(n-x) = 0$

$\therefore x = 0, n$

$\therefore$ there is a stationary point at $(n,\ n^n e^{-n})$

$f'(x) = x^{n-1}e^{-x}(n-x)$

NB $x^{n-1}e^{-x} > 0$ for $x > 0$

Noting that $x^{n-1}e^{-x} > 0$, then the sign of $f'(x)$ is determined by $(n - x)$

$x < n$ (Take $x = n - 1$): $f'(x) > 0$

$x > n$ (Take $x = n + 1$): $f'(x) < 0$

$\therefore (n,\ n^n e^{-n})$ is a maximum turning point.

Comment:
Many students didn't read the question properly and so didn't know what the question demanded of them. Students had to give the coordinates and justify why it was a maximum to get full marks. Some students found $f''(x) = e^{-x}x^{n-2}[x^2 - 2nx + n(n-1)]$.

(iii) Use part (ii) to assist in explaining why $\lim_{x\to\infty} x^n e^{-x} = 0$ for $n \geq 0$. **1**

First, for $x > 0$, $0 < x^n e^{-x} \leq n^n e^{-n}$ i.e. there is no x-intercept for $x > 0$.

Since $x = n$ is the only maximum stationary point, then for $x > n$, $x^n e^{-x}$ is decreasing.

So as $x \to \infty, x^n e^{-x} \to 0$.

Comment:
The purpose of this question was to help students in the part (iv), but to answer it, students had to explain why $\lim_{x\to\infty} x^n e^{-x} = 0$ using part (ii), not write anything they could.

Students who just quoted standard results about exponential functions or stated that e^{-x} "dominates" x^n got no credit.

Question 9 (continued)

(c) (iv) The Gamma Function, $\Gamma(n)$, is defined as $\Gamma(n) = \int_0^{\infty} x^{n-1}e^{-x}\,dx$. **2**

Show that $\Gamma(n) = (n-1)\Gamma(n-1)$.

(Note that $\int_0^{\infty} x^{n-1}e^{-x}\,dx = \lim_{M\to\infty}\int_0^{M} x^{n-1}e^{-x}\,dx$.)

$$\Gamma(n) = \int_0^{\infty} \underbrace{x^{n-1}}_{u}\underbrace{e^{-x}}_{v'}\,dx$$

$$u = x^{n-1} \Rightarrow u' = (n-1)x^{n-2}$$

$$v' = e^{-x} \Rightarrow v = -e^{-x}$$

$$\Gamma(n) = \int_0^{\infty} \underbrace{x^{n-1}}_{u}\underbrace{e^{-x}}_{v'}\,dx$$

$$= uv - \int_0^{\infty} u'v\,dx$$

$$= \left[-x^{n-1}e^{-x}\right]_0^{\infty} - \int_0^{\infty}(n-1)x^{n-2}(-e^{-x})\,dx$$

$$= 0 + (n-1)\int_0^{\infty} x^{n-2}e^{-x}\,dx$$

$$= (n-1)\Gamma(n-1)$$

From (iii): $\lim_{x\to\infty} x^n e^{-x} = \lim_{x\to\infty} x^{n-1}e^{-x} = 0$

Comment:

The note given above was to explain how to evaluate an integral with no finite upper limit. It was not the intention that students would write a series of limit statements.

Some students tried to take the approach of $\Gamma(n) = \int_0^{\infty} e^{-x}\,d\left(\frac{x^n}{n}\right)$

This way can work:

$$\Gamma(n) = \int_0^{\infty} e^{-x}\,d\left(\tfrac{x^n}{n}\right)$$

$$= \left[\tfrac{x^n}{n}e^{-x}\right]_0^{\infty} - \frac{1}{n}\int_0^{\infty} -e^{-x}x^n\,dx$$

$$= \frac{1}{n}\int_0^{\infty} e^{-x}x^n\,dx$$

$$= \frac{1}{n}\Gamma(n+1)$$

$$\therefore \Gamma(n) = \frac{1}{n}\Gamma(n+1) \Rightarrow \Gamma(n+1) = n\Gamma(n)$$

Let $m = n + 1$ and then $\Gamma(m) = (m-1)\Gamma(m-1)$ as required

Question 9 (continued)

(c) (v) Show that $\Gamma(n)=(n-1)!$ for integral $n > 0$. **2**
You are NOT required to use mathematical induction.

From (iv)
$$\begin{aligned}\Gamma(n) &= (n-1)\Gamma(n-1)\\ &= (n-1)(n-2)\Gamma(n-2)\\ &= (n-1)(n-2)(n-3)\Gamma(n-3)\\ &= (n-1)(n-2)(n-3)\times\ldots\times 2\times\Gamma(2)\\ &= (n-1)(n-2)(n-3)\times\ldots\times 1\times\Gamma(1)\\ &= (n-1)!\times\Gamma(1)\end{aligned}$$

$$\begin{aligned}\text{Now } \Gamma(1) &= \int_0^\infty e^{-x}\,dx\\ &= \left[-e^{-x}\right]_0^\infty\\ &= 0-(-1)\\ &= 1\end{aligned}$$

$$\therefore \Gamma(n)=(n-1)!$$

Comment:

Students who didn't evaluate $\Gamma(1)$ properly did not get full marks.

Some students made the mistake of trying to evaluate $\Gamma(0)$.

Question 9 (continued)

(d) (i) Show that $\int \sin^n x\,dx = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx$. **4**

$$\text{Let } I_n = \int \sin^n x\,dx$$

$$= \int \underbrace{\sin^{n-1}x}_{u}\underbrace{\sin x}_{v'}\,dx$$

$$= \cos x \sin^{n-1} x - \int \cos x \times (n-1)\sin^{n-2} x \cos x\,dx$$

$$= \cos x \sin^{n-1} x - (n-1)\int \cos^2 x \sin^{n-2} x\,dx$$

$$= \cos x \sin^{n-1} x - (n-1)\int (1-\sin^2 x)\sin^{n-2} x\,dx$$

$$= \cos x \sin^{n-1} x - (n-1)\int (\sin^{n-2} x - \sin^n x)\,dx$$

$$= \cos x \sin^{n-1} x - (n-1)(I_{n-2} - I_n)$$

$$I_n + (n-1)I_n = \cos x \sin^{n-1} x - (n-1)I_{n-2}$$

$$I_n = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

Comment:

The signs caused problems for some students, though most students knew this standard textbook problem. As well some students tried to start with

$$I_n = \int \sin^{n-2} x \sin^2 x\,dx$$

<u>It can work:</u>

$$I_n = \int \sin^{n-2} x(1-\cos^2 x)\,dx \qquad = I_{n-2} - \int \sin^{n-2} x \cos^2 x\,dx$$

$$= I_{n-2} - \int \underbrace{\cos x}_{u}.\underbrace{\cos x \sin^{n-2} x}_{v'}\,dx$$

$$= I_{n-2} - \left[\cos x \times \tfrac{1}{n-1}\sin^{n-1} x - \int \tfrac{1}{n-1}\sin^{n-1} x(-\sin x)\,dx\right]$$

$$I_n = I_{n-2} - \cos x \times \tfrac{1}{n-1}\sin^{n-1} x - \tfrac{1}{n-1} I_n$$

$$(n-1)I_n = (n-1)I_{n-2} - \cos x \sin^{n-1} x - I_n$$

$$nI_n = (n-1)I_{n-2} - \cos x \sin^{n-1} x$$

And it now finishes like the one above.

Question 9 (continued)

(d) (ii) Evaluate $\int_0^{\frac{\pi}{3}} \sin^3 x \, dx$. **2**

$$\begin{aligned}\int_0^{\frac{\pi}{3}} \sin^3 x \, dx &= \left[-\frac{\sin^2 x \cos x}{3}\right]_0^{\frac{\pi}{3}} + \frac{2}{3}\int_0^{\frac{\pi}{3}} \sin x \, dx \\ &= -\frac{1}{3}\left(\frac{3}{4}\times\frac{1}{2}-0\right)+\frac{2}{3}\left[-\cos x\right]_0^{\frac{\pi}{3}} \\ &= -\frac{1}{8}+\frac{2}{3}\left(-\frac{1}{2}+1\right) \\ &= -\frac{1}{8}+\frac{1}{3} \\ &= \frac{5}{24}\end{aligned}$$

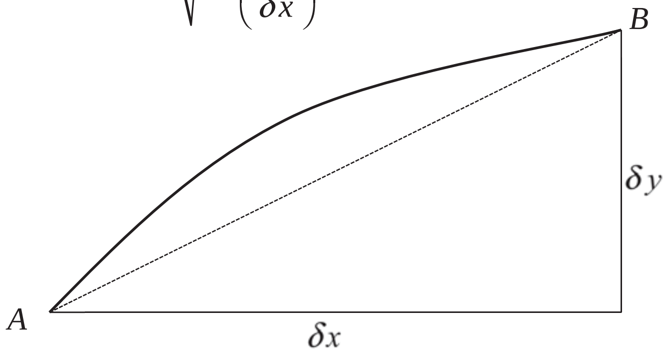
Comment:

For most this was a very simple problem.
It was surprising the number of students who couldn't use the formula developed in part (i) or chose not to use it and do this problem as if it was a stand alone problem.

Also several students need to brush up on their trigonometric results e.g. $\cos 0 = 1$.

Question 9 (continued)

(e) (i) Show, using Pythagoras' Theorem, that an approximation to the length of arc AB is $\sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2}\ \delta x$. **2**



For small δx and δy, $AB \doteqdot$ arc AB

By Pythagoras' Theorem,

$$AB=\sqrt{(\delta x)^2+(\delta y)^2}$$
$$=\sqrt{\frac{(\delta x)^2+(\delta y)^2}{(\delta x)^2}\times(\delta x)^2}$$
$$=\sqrt{\frac{(\delta x)^2}{(\delta x)^2}+\frac{(\delta y)^2}{(\delta x)^2}}\ \ \delta x$$
$$=\sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2}\ \ \delta x$$

Comment:

There was 2 marks allocated for this problem, and given the trivial nature of coming up with $AB=\sqrt{(\delta x)^2+(\delta y)^2}$, students had to actually show something.

A student who just quoted $AB=\sqrt{(\delta x)^2+(\delta y)^2}$ was awarded $\frac{1}{2}$ mark.
This also applied to students who only wrote:

$$AB=\sqrt{(\delta x)^2+(\delta y)^2}$$
$$=\sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2}\ \delta x$$

When asked to "Show that … ", students MUST show more information/detail than what was provided for them.

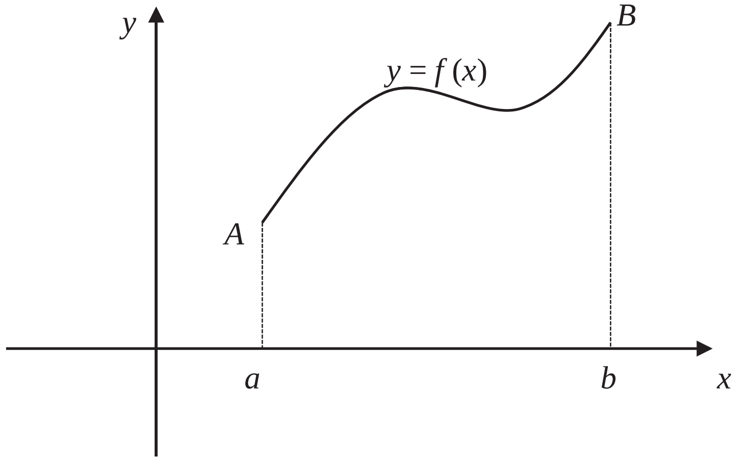
Students who started with the RHS fared better.

Question 9 (continued)

(e) (ii) The length of the arc AB is given by

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2}\,\delta x = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Verify, using integration, that the length of the semicircle $y=\sqrt{4-x^2}$ is 2π.



$$y=\sqrt{4-x^2}=(4-x^2)^{\frac{1}{2}}$$

$$\frac{dy}{dx}=\frac{1}{2}(4-x^2)^{-\frac{1}{2}}\times(-2x)=-\frac{x}{\sqrt{4-x^2}}$$

$$\left(\frac{dy}{dx}\right)^2=\frac{x^2}{4-x^2}$$

$$1+\left(\frac{dy}{dx}\right)^2=1+\frac{x^2}{4-x^2}=\frac{4}{4-x^2}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\frac{2}{\sqrt{4-x^2}}$$

$$\begin{aligned}\text{Arc length} &= \int_{-2}^{2}\frac{2}{\sqrt{4-x^2}}\,dx\\ &= 4\int_0^2\frac{1}{\sqrt{4-x^2}}\,dx\\ &= 4\left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2\\ &= 4\left(\frac{\pi}{2}-0\right)\\ &= 2\pi\end{aligned}$$

Comment:

If a student could get to $\int_{-2}^{2}\frac{2}{\sqrt{4-x^2}}\,dx$ then they were generally successful.
Though many students didn't recognise this as an integral in their Reference Sheet.
Students who misread the question and evaluated $\int_{-2}^{2}\sqrt{4-x^2}\,dx$ could only earn 1 mark.

End of solutions